varied continuously from zero.

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ON THE LOSS OF STABILITY OF THE SHAPE OF AN IDEALLY FLEXIBLE STRING

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The equations of the dynamics of an ideally flexible string were solved relative to the curvature and torsion of its shape in [1], and the characteristic wave propagation velocities of these parameters were found. A connection between the characteristic velocities and the loss of stability of the shape is established herein, which is identified with the loss in correctness of formulating problems with initial conditions.

We understand an ideally flexible string to be a material line which does not resist a change in shape, i.e. in curvature Ω_3 and torsion Ω_1.

Let the unperturbed motion of the string be characterized by the equations

$$\Omega_i^0 = \Omega_i^0(s, t) \qquad (i = 1,3)$$

Here s is the arc coordinate, and t the time.

Let us give some small deviations $\varepsilon_i^0(s)$ from the unperturbed values to the curvature and torsion by demanding that these deviations satisfy appropriate boundary conditions. The perturbed motion of the string then becomes

$$\Omega_i = \Omega_i^0(s, t) + \varepsilon_i(s, t) \qquad (i = 1,3)$$

In some domain D $(0 \leqslant s \leqslant s_1,\ 0 \leqslant t \leqslant t_1)$ let the following inequalities hold

$$\max|\Omega_i - \Omega_i^0| < \delta, \qquad \max|\varepsilon_i| < \nu \tag{1}$$

Let us consider the string shape unstable in the domain D if

$$\delta \to \delta_0 > 0 \text{ for } \nu \to 0. \tag{2}$$

Let us consider an arbitrary system of equations with constant coefficients

$$\sum_{j=1}^{n}\left(a_{ij}\frac{\partial q_j}{\partial t} + b_{ij}\frac{\partial q_j}{\partial s} + c_{ij}q_j\right) = 0 \tag{3}$$

Let the initial conditions be

$$q_m^\circ = \frac{1}{\lambda_1}e^{-\lambda_1 s i}, \quad \lambda_1 > 0; \quad q_j^\circ = 0, \quad (j = 1, 2, \ldots, n \neq m) \quad \text{for} \quad t = 0;\ 0 \leqslant s \leqslant s_1 \tag{4}$$

Let us assume that the relationships (4) satisfy the boundary conditions of Eq. (3).

Dispensing with the specific form of these boundary conditions, we assume that the series of appropriate eigenvalues $\lambda_1^1, \lambda_1^2, \ldots$ etc. has no upper bound.

The solution of (3) can be written as

$$q_j = \sum_{k=1}^{n} A_{jk} e^{\lambda_1(\lambda_k t - s)i} \qquad (j = 1, \ldots, n) \tag{5}$$

Here the λ_k are roots of the characteristic equation

$$\det\left(a_{lj}\lambda_k - b_{lj} - \frac{c_{lj}}{\lambda_1}\right) = 0 \tag{6}$$

and the A_{jk} are determined from the system

$$\sum_{j=1}^{n}\left(a_{lj}\lambda_k - b_{lj} - \frac{c_{lj}}{\lambda_1}\right) A_{jk} = 0 \qquad (k = 1, \ldots, n) \tag{7}$$

to the accuracy of arbitary constants $C_1, \ldots, C_k$. Let us first assume that all the roots λ_k are distinct. Then

$$A_{jk} = B_{jk} C_k, \qquad \det(B_{jk}) \neq 0 \tag{8}$$

where the coefficients B_{jk} are determined just by the coefficients of the system (7) and the values of λ_k. For every fixed value k_+ there is evidently a value j_+ such that

$$B_{j_+k_+} \neq 0$$

To determine the C_k we have the system

$$q_j^\circ = \sum_{k=1}^{n} B_{jk} C_k \tag{9}$$

Here

$$q_j^0 = q_j \quad \text{for } t = 0,\ s = 0$$

Let us assume that

$$q_{j=m}^\circ = \frac{1}{\lambda_1}, \quad q_{j\neq m}^\circ = 0 \tag{10}$$

Then

$$C_k = \frac{1}{\lambda_1}\alpha_k, \quad \left\| \begin{matrix} \alpha_1 \\ \cdots \\ \alpha_k \\ \cdots \\ \alpha_n \end{matrix} \right\| = \| B_{jk} \|^{-1} \left\| \begin{matrix} 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{matrix} \right\| \tag{11}$$

Since the matrix (B_{jk}) is nonsingular, the coefficients α_k cannot equal zero simultaneously. Taking account of (9) and (11), the relationship (5) becomes

$$q_j = \frac{1}{\lambda_1}\sum_{k=1}^{n} B_{jk}\alpha_k e^{\lambda_1(\lambda_k t - s)i}, \qquad (j = 1, \ldots, n) \tag{12}$$

Besides the relationships (6) and (7), let us consider the auxiliary relationships

$$\det(a_{lj}\lambda^* - b_{lj}) = 0, \qquad \sum_{j=1}^{n}(a_{lj}\lambda_k^* - b_{lj}) A_{jk}^* = 0 \tag{13}$$

Evidently the λ_k^* and A_{jk}^* determined from (13) are independent of λ_1, so the coefficients B_{jk}^* and α_k^* determined from (7), (8), (11) by means of λ_k^* and A_{jk}^* are also independent of λ_1. Let us note here that the λ_k^* are characteristic velocities of the waves being propagated.

Let $\lambda_1 \to \infty$. Then

$$\lambda_k \to \lambda_k^*, \quad A_{jk} \to A_{jk}^*, \quad B_{jk} \to B_{jk}^*, \quad \alpha_k \to \alpha_k^*, \quad (j = 1, \ldots, n) \tag{14}$$

and for sufficiently large values of λ_1 the relationship (12) can be written as

$$q_j = \frac{1}{\lambda_1} \sum_{k=1}^{n} (B_{jk}^* \alpha_k^* + \varepsilon_k) e^{\lambda_1 [(\lambda_k^* + \zeta_k) t - s] i} \qquad (j = 1, \ldots, n) \tag{15}$$

where ε_k, ζ_k are small positive numbers and

$$\varepsilon_k, \zeta_k \to 0 \quad \text{for} \quad \lambda_1 \to \infty \tag{16}$$

Let us assume that there are complex numbers among the λ_k^*. The number of complex roots is even because of the reality of the coefficients a_{ij}, b_{ij}, and there is a $\lambda_{k_+}^*$ in each pair of these roots for which $\mathrm{Jms}\lambda_{k_+}^* < 0$.

Let us select the number m in (10) in such a way that the inequality $\alpha_{k_+} \neq 0$ would hold in conformity with (11). It is easy to see that there is always such a possibility since the matrix (B_{jk}) is nonsingular. Indeed, let us consider the k_+-th row of matrix $(B_{jk})^{-1}$. Among the elements of this row there is certainly a nonzero element. Let this element correspond to the pth column. Selecting $m = p$, we arrive at the nonzero value α_{k_+}. It has been noted above that for each k_+ there is at least a value j_+ such that $B_{j_+ k_+} \neq 0$.

From (15) let us select a solution for q_{j_+} which can be written thus:

$$q_{j_+} = \frac{1}{\lambda_1} \left\{ (B_{j_+ k_+}^* \alpha_{k_+}^* + \varepsilon_k) e^{\lambda_1 [(\lambda_{k_+}^* + \zeta_{k_+}) t - s] i} + \sum_{k \neq k_+} (B_{j_+ k}^* \alpha_k^* + \varepsilon_k) e^{\lambda_1 [(\lambda_k^* + \zeta_k) t - s] i} \right\} \tag{17}$$

Letting $\lambda_1 \to \infty$, we obtain

$$q_{j_+} \to \infty \quad \text{for} \quad q_j^\circ \to 0 \tag{18}$$

where the relationships (18) hold for any s_1 and t_1, i.e. for any domains D.

The reasoning is not changed if there are multiple roots among the λ_k. The coefficients B_{jk} will hence be polynomials in t, which does not affect the result (8).

The formulated result has been obtained under "artificial" initial conditions determined from the relationships (13) at $t = 0$, namely

$$q_j (t = 0) = \frac{1}{\lambda_1} \sum_{k=1}^{n} B_{jk} \alpha_k e^{-\lambda_k i} \tag{19}$$

Let the initial conditions of the system (3) be arbitrary (but satisfy the boundary conditions)

$$q_j (t = 0) = q_j^* (s) \qquad (j = 1, \ldots, n) \tag{20}$$

and let its solution become

$$q_j = q_j^* (s, t) \qquad (j = 1, \ldots, n) \tag{21}$$

Let us append the initial conditions (19) to the initial conditions (20). Selecting λ_1 sufficiently large, the changes in the initial conditions (20) can be made so small that they will not emerge beyond the limits of accuracy of their assignment.

However, the solution of the original system (3) on the basis of the superposition principle becomes

$$q_j = q_j^* (s, t) + \frac{1}{\lambda_1} \sum_{j=1}^{n} B_{jk} \alpha_k e^{\lambda_1 (\lambda_k t - s) i} \tag{22}$$

and possesses the same property as the solution (14) for $\lambda_1 \to \infty$. Thus, the presence of complex elastic wave propagation velocities which occur when the initial system will be ultra-hyperbolic or elliptic, will be a sufficient condition for instability, in the class of differentiable functions, of the solution of the system (3) in problems with initial conditions. This result is in good agreement with the Hadamard [2] example, and can be considered its generalization.

Let the original system (3) be quasilinear, i.e. a_{ej}, b_{ej}, c_{ej} depend on q_j, t, s. Let us assume these dependencies are given by analytic functions. Let us linearize the system relative to some unperturbed values $q_j^0(s, t)$, and let us linearize the obtained coefficients of the linearized system $a_{ej}^0(s, t)$, $b_{ej}^0(s, t)$, $c_{ej}^0(s, t)$ once more relative to the fixed values $s_0 = 0$, $t_0 = 0$.

We therefore arrive at a linear system with constant coefficients of the type in (3), which will be "close" to the original quasilinear system for "sufficiently small" t, s and $\Delta q_j = q_j - q_j^0$. The results obtained above on the loss of stability remain valid even for a quasilinear system in the same domain with the sole difference being that a weaker assertion should here be satisfied, namely: from $\Delta q_j\ (t = 0) \to 0$ follows $\Delta q_j \geqslant \delta_0 > 0$, where δ_0 is a sufficiently small fixed number since the linearized system may differ substantially from the original for $\Delta q_j \to \infty$.

Indeed, if $\Delta q_j \to 0$ were to follow for arbitrarily small t and s from $\Delta q_j\ (t = 0) \to 0$ for the original system, then this same result would hold in the linearized system.

Therefore, if the system (3) is quasilinear and in some sufficiently small neighborhood of the values t_0, s_0, q_0 will be ultra-hyperbolic or elliptic, then its solutions in problems with initial conditions are unstable in the class of differentiable functions. It is hence certainly assumed that the existence and uniqueness of these solutions hold under appropriate boundary conditions.

It has been shown in [1] that two kinds of characteristic wave propagation velocities (curvature and torsion waves) hold in an inextensible ideally flexible string

$$\lambda_1 = \pm\left(\frac{T}{\rho}\right)^{1/2}, \quad \lambda_2 = \pm\left[\frac{T}{\rho} + \frac{1}{\rho\Omega_3}\left(F_2 + \frac{\partial F_3}{\partial v_2} v_3 - \frac{\partial F_3}{\partial v_3} v_2\right)\right]^{1/2} \tag{23}$$

Here T is the tension, ρ the linear density, Ω_3 the curvature of the shape, $F_1, F_2, F_3, v_1, v_2, v_3$ the projections of resistive forces, and the velocities on the axes of the natural trihedron of the string, respectively.

According to the results obtained above, the stability of the string shape is lost upon compliance with one of the two conditions

$$T < 0, \quad T + \frac{1}{\Omega_3}\left(F_2 + \frac{\partial F_3}{\partial v_2} v_3 - \frac{\partial F_3}{\partial v_3} v_2\right) < 0 \tag{24}$$

The first inequality indicates that the shape of a compressed flexible string is unstable. This deduction is in good agreement with everyday practical observations. Indeed, although the compressed state of a flexible string does not contradict any of the laws of mechanics, this state is practically unrealizable because a compressed flexible string is unstable.

However, it follows from the second inequality in (24) that instability can set in in the string not only under compression; it can even occur in an extended string if the parenthesis in (24) is negative and the curvature Ω_3 is sufficiently small (for example, near the inflection point). This result is less evident and not provided for directly although the loss of stability should occur here physically with the same intensity as in the first case, with the sole difference that the cause of the loss of stability in the first case will be degeneration of the curvature waves, and of the torsion waves in the second case.

It is known [1] that a third system of waves (tension waves) is manifest in an extensible string. It is easy to see that the propagation velocities of these waves will not be subject to degeneration for any of the string parameters. Therefore, the stability of the shape of extensible strings is lost upon compliance with the same inequalities (24), as for an inextensible string.

Let the string move over a surface. In conformity with the results obtained in [1], two cases should be distinguished, namely: (a) the string will coincide with the asymptotic curve of the surface, and (b) the string will not coincide with the asymptotic curve of the surface. Only curvature waves being propagated with velocity λ_1 will occur in the first case, and the loss of stability of the shape occurs at $T < 0$. An illustration for this case is the motion of a string over a plane accomplishing a bilateral connection. In the second case there occur torsion waves propagated with the velocity

$$\lambda_2' = \pm \sqrt{\rho^{-1}(T + A/B)}, \qquad B = \Omega_3 [1 + k(p_3 \sin\delta + p_2 \cos\delta)]$$

$$A = \cos\delta \left[(\sin\delta + kp_3)\left(F_3 + \frac{\partial F_2}{\partial v_2} v_3 - \frac{\partial F_2}{\partial v_3} v_2\right) + (\cos\delta + kp_2) \times \right.$$
$$\left. \times \left(F_2 + \frac{\partial F_3}{\partial v_2} v_3 - \frac{\partial F_3}{\partial v_3} v_2\right)\right] \tag{25}$$

Here p_1, p_2, p_3 are the direction cosines of the friction vector in the axes of the natural trihedron, k is the friction coefficient of the string on the surface, and δ is the angle of geodesic declination of the string on the surface.

In this case the condition for loss of stability becomes

$$T + A/B < 0 \tag{26}$$

The results presented above arouse interest at those points of the string at which regeneration of the hyperbolic system of dynamics equations into an ultra-hyperbolic or elliptic system occurs, i.e. at those points of the string at which equalities hold in place of the inequalities (24) or (26). The neighborhood of these points should be subjected to special investigation. One such investigation has been performed in [3], where it has been shown that an abrupt rise in velocity (a flick of the whip), connected with the loss in stability of the shape near the point $T = 0$, occurs near the free end of the string (for $T = 0$).

In conclusion, let us consider the singularities of the formulated instability condition (2) in rather more detail. According to this condition, instability is manifested in an arbitrarily small time interval as contrasted to Liapunov instability, which is disclosed as $t \to \infty$. In this sense, the considered instability will be stronger than Liapunov instability.

Also essential is the fact that the criteria of the introduced instability depend only on the coefficients of the highest derivatives in the appropriate dynamics equations (by which the kind of system is determined), and is independent of not only the remaining coefficients, but also of the boundary conditions. The introduced instability therefore reflects quite general dynamic properties of the string, and can be called absolute instability. It is clear that absolute instability is not realizable, in principle, in systems with a finite number of degrees of freedom. Up to now this instability has not been detected in three-dimensional continuous systems, since waves carrying changes in the internal geometry parameters (tension, shear, pressure waves), do not degenerate for any values of the parameters (even for nonlinear strain characteristics). And absolute instability can occur just in one- or two-dimensional continuous systems, where "transverse" waves or waves carrying changes in the shape parameters, i.e. the external geomery, exist.

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ON THE UNLOADING WAVE IN MATERIALS WITH DELAYED YIELDING

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The question of the existence of an unloading wave in the case of one-dimensional wave propagation in a semi-infinite rod of material with delayed yielding is solved herein. Unloading conditions are formulated here, and an analytic method to obtain an expression for the initial velocity of the unloading wave is expounded.

1. Unloading condition. The dependence $\sigma \sim \varepsilon \sim t$ is taken in the form $\sigma = F(\varepsilon, t - x/a_0)$ in [1] for problems of active longitudinal wave propagation in a rod from material with delayed yielding. In particular, the solution is investigated for the law

$$\sigma = E\varepsilon, \quad |\varepsilon| \leqslant \varepsilon_s, \quad \sigma = E_1\varepsilon + (E - E_1)\,\varepsilon_s(t - x/a_0), \quad |\varepsilon| > \varepsilon_s \tag{1.1}$$

which corresponds to linear hardening upon instantaneous loading. Here ε_s is a monotonely decreasing function of its argument. Henceforth considering only the tension case ($\sigma \geqslant 0$, $\varepsilon \geqslant 0$), let us note that the requirement for a growth in stress in the cross section, in particular, of loading on the endface of a semi-infinite rod is not necessary. Indeed, defining the plastic deformation as $\varepsilon^p = \varepsilon - \sigma / E$, as is customary, we see that the transition to passive strain is determined by the requirement

$$E\frac{\partial\varepsilon}{\partial t} - \frac{\partial\sigma}{\partial t} \leqslant 0 \tag{1.2}$$

By using (1.1) this condition can be represented in one of the two forms

$$\frac{\partial\varepsilon}{\partial t} \leqslant \varepsilon_S'\left(t - \frac{x}{a_0}\right) \quad \text{or} \quad \frac{\partial\sigma}{\partial t} \leqslant E\varepsilon_S'\left(t - \frac{x}{a_0}\right) \tag{1.3}$$

The limiting case of "neutral" loading investigated in [1] (domain 2 in Fig. 1, and Formula (13) in the mentioned paper) corresponds to the equality sign in (1.3). In particular, unloading at the end of the rod starts at time $t = t_0$ if the applied stress $\varphi(t) = \sigma(0, t)$ satisfies the condition

$$\varphi'(t) \leqslant E\varepsilon_s'(t) \quad \text{for} \quad t \geqslant t_0 \tag{1.4}$$

2. Unloading wave. Let τ be the time of origination of plastic deformation at the end $x = 0$ of a semi-infinite rod, and let condition (1.4) be satisfied from the time $t = t_0 \geqslant \tau$. Let us show that the boundary between the active and passive strain domains is $t = f(x)$ in the plane of the characteristics (x, t), i.e. the unloading wave has a finite propagation velocity $b = 1/f'(x)$ satisfying the condition

$$a_1 \leqslant b \leqslant a_0, \qquad a_0 = \sqrt{E/\rho}, \qquad a_1 = \sqrt{E_1/\rho} \tag{2.1}$$

Here a_0, a_1 are propagation velocities of the longitudinal elastic and plastic waves, respectively. For the unloading domain we take the connection between the stress and